Indicial Coefficients for a Cropped Delta Wing in Incompressible Flow

Valter J.E. Stark*
Saab Scania AB, Linköping, Sweden

Indicial coefficients and corresponding deficiency functions have been calculated for typical deflection modes of a cropped delta wing with trailing-edge flaps in incompressible flow by an approximate doublet element method for arbitrary motion. The normalized deficiency functions for the modes considered were found to be closely approximated by a single one-parameter function. The parameter, a characteristic time, was found to have the same value for this wing as for a rectangular wing of aspect ratio 3 when referred to semispan divided by freestream speed. Laplace transformation of the one-parameter function yields a generalized Theodorsen function.

Nomenclature

$A_{m,n}(p)$	= aerodynamic transfer function
b	= semispan referred to L
$C(-ip)$	= Theodorsen's function
$C^r_{m,n}(t)$	= deficiency function
$D^r_{m,n}$	= apparent mass coefficient
$E_1(p)$	= exponential integral
$h_n(x,y)$	= deflection mode referred to L
$K_{m,n}(t)$	= generalized aerodynamic force coefficient referred to qS
$K^r_{m,n}(t)$	= indicial aerodynamic coefficient
L	= reference length, root semichord
$L\{f,p\}$	= Laplace transform of $f(t)$
M	= Mach number
p	= Laplace transform parameter referred to U/L
$\Delta p_n, \Delta p^r_n$	= pressure jump and indicial pressure jump, respectively, for nth mode referred to q
q	= dynamic pressure, $\rho U^2/2$
S	= reference area, wing area
T	= characteristic time referred to L/U
t	= time referred to L/U
U	= freestream speed
x,y,z	= streamwise, spanwise, and normal coordinates, respectively, referred to L
$\Phi(t)$	= Wagner's function
ϕ^r_n	= indicial potential for nth mode referred to UL
$\Delta\phi^r_n$	= jump in indicial potential for nth mode referred to UL
$\varphi(t)$	= approximate normalized deficiency function

Introduction

IN flutter programs wherein the nonlinear eigenvalue problem is solved,[1] it is desirable to use comparatively simple approximations for the aerodynamic transfer functions. Attempts to find such approximations have shown that a single one-parameter function[2] may be sufficient for finite wings in incompressible flow.

The function studied is of the same class as employed by Garrick.[3] He proposed the simple approximation $1-(1/2)/(1+t/T)$ for the Wagner function $\Phi(t)$, which yields the approximation $1-(1/2)pTE_1(pT)\exp(pT)$ for the Theodorsen function $C(-ip)$. E_1 is the exponential integral and T a characteristic time equal to 4 if t is measured in semichords traveled. Since the approximation for Φ is accurate (deviates less than 2%) and behaves in the correct way for $t\to\infty$, the approximation for $C(-ip)$ is accurate and behaves in the correct way for $p\to 0$. The latter approximation has a discontinuity of the same kind as $C(-ip)$ on the negative real axis of the p plane.

In the case of a finite wing, indicial aerodynamic coefficients are required. Similar to the coefficients calculated by Lomax et al.,[4] those calculated herein correspond to a step change in the normal velocity, while those considered by Milne[5] and Vepa[6] correspond to an impulsive change. It is preferred, however, to work herein with the deficiency function.[2] These correspond to $1-\Phi$ and result from subtracting the indicial coefficients from their limits for $t\to\infty$.

For subsonic flow, it may be assumed that the deficiency functions can be approximated by a sum of terms of the form $a_n/(1+t/t_n)^k$, where t_n and k are given parameters and a_n coefficients to be determined. The sum may have three or more terms, since at least three characteristic times t_n are required for compressible flow. The first characteristic time t_1 may correspond to the development of the vortex sheet behind the wing, and t_2 and t_3 may be the times required for sound waves to propagate in the upstream and downstream directions a distance equal to the streamwise extent of the wing. If c is this distance, a the speed of sound, and M the freestream Mach number, these times are given by $t_2=(c/a)/|1-M|$ and $t_3=(c/a)/(1+M)$. For $a\to\infty$, they reduce to zero so that only one term remains for incompressible flow.

The remaining term appears to be sufficient for $M=0$. In a previous investigation,[2] in which deficiency functions were calculated for a rectangular wing of aspect ratio 3 in incompressible flow, it was found, for example, that all calculated values for normalized deficiency functions for different deflection modes were closely approximated by a single function. This function is $\varphi(t)=1/(1+t/T)^3$, where T was determined in such a way that the close agreement was obtained. The exponent 3 is appropriate for three-dimensional flow,[5] while replacing it by unity would yield a function with behavior proper for two-dimensional flow.

It is assumed that this function can be used for all finite wings in incompressible flow and that a value that yields a close approximation for all important modes can be determined for T for each wing. The result of this investigation supports the assumption.

Approximating the generalized aerodynamic coefficients by means of φ and applying Laplace transformation results in an expression of the form $A^0+A^1p+A^2p^2-(B^0+B^1p)pL\times\{\varphi,p\}$ for the aerodynamic transfer functions. A suitable

Received March 21, 1985; revision received Nov. 29, 1985.

*Research Scientist. Member AIAA.

value for T can probably be selected without calculation, but the constants A^0, A^1, A^2, B^0, and B^1 must be determined, for example, by the method of least squares. The values of the aerodynamic transfer functions which are then needed can be calculated by an oscillating-surface program.

Over 40 years ago, R.T. Jones[7] and W.P. Jones[8] proposed approximations for the Wagner function and in 1976 Vepa[6] proposed corresponding approximations for finite wings. The approximations of Jones and Jones contain exponential terms $a_n \exp(-t/t_n)$ which transform into terms of the form $a_n p t_n/(1+pt_n)$ in the p domain. When writing the sum of these terms as a ratio of two polynomials, it can be seen that the rational approximations of Vepa are equivalent to the Jones' approximations. The name Padé approximant used for these approximations is appropriate only if the unknown parameters (a_n and t_n) are determined in a certain way.[9]

If Jones' or Vepa's approximations are used for the aerodynamic transfer functions, the determinant of the equations of motion in the p domain has certain zeros, called aerodynamic lag roots.[10] These zeros, which are due to the poles $p=-1/t_n$ of the approximation, do not appear if the approximation proposed herein is employed.

Indicial Coefficients

In the following, all quantities are dimensionless and referred to typical reference quantities. Lengths are referred to the root-semichord L, velocities to the freestream speed U, times to L/U, pressures to the freestream dynamic pressure $\rho U^2/2$, and velocity potentials to UL.

It is assumed that the perturbation pressure can be calculated by the linearized theory and the deflection of the wing can be described by the following equation:

$$z=\sum_{n=1}^{n_s} h_n(x,y)q_n(t) \tag{1}$$

where the functions $h_n(x,y)$ are given deflection modes and $q_n(t)$ undetermined generalized coordinates. The former depend only on the coordinates x and y (in the freestream and spanwise directions, respectively) and the latter only on the time t.

The perturbation pressure and perturbation velocity potential can be decomposed into components p_n and ϕ_n which correspond to the nth term in Eq. (1). Since p_n is related to ϕ_n through the linearized Bernoulli equation, the pressure jump Δp_n and potential jump $\Delta\phi_n$ across the wing or wake must satisfy

$$\Delta p_n=-2\left(\frac{\partial\Delta\phi_n}{\partial x}+\frac{\partial\Delta\phi_n}{\partial t}\right) \tag{2}$$

The potential ϕ_n is determined by the wave equation (referred to the moving coordinates x,y,z), radiation condition, wake condition, Kutta condition, and boundary condition on the wing:

$$\frac{\partial\phi_n}{\partial z}=\left(\frac{\partial h_n}{\partial x}\right)q_n+h_n\dot{q}_n \tag{3}$$

where $\dot{q}_n=\mathrm{d}q_n/\mathrm{d}t$.

The generalized aerodynamic forces for arbitrary motion are represented by the dimensionless coefficients $K_{m,n}(t)$, defined by

$$K_{m,n}(t)=\frac{1}{S}\int\!\!\int h_m\Delta p_n \mathrm{d}s \tag{4}$$

where S is a reference area, usually the wing area, and dS a surface element measured by the same unit as S.

Two indicial aerodynamic coefficients, $K^1_{m,n}(t)$ and $K^2_{m,n}(t)$, can be used to express the aerodynamic forces for arbitrary motion in terms of the generalized coordinates. They are associated with the indicial potentials ϕ^1_n and ϕ^2_n defined[2] by the same differential equation and conditions as ϕ_n, but the prescribed normal velocity is different. The boundary condition for ϕ^r_n on the wing reads

$$\begin{aligned}\frac{\partial\phi^r_n}{\partial z}&=\left(\frac{\partial h_n}{\partial x}\right)H(t), \qquad r=1\\ &=h_nH(t), \qquad r=2\end{aligned} \tag{5}$$

where $H(t)$ is the Heaviside unit step function. The indicial pressure jumps are defined by

$$\Delta p^r_n=-2\left(\frac{\partial\Delta\phi^r_n}{\partial x}+\frac{\partial\Delta\phi^r_n}{\partial t}\right) \tag{6}$$

and the indicial coefficients by

$$K^r_{m,n}(t)=\frac{1}{S}\int\!\!\int h_m\Delta p^r_n \mathrm{d}S \tag{7}$$

By means of these coefficients, the generalized aerodynamic forces for arbitrary motion can be written as

$$K_{m,n}(t)=\int_0^t K^1_{m,n}(t-\tau)\dot{q}_n(\tau)\mathrm{d}\tau+\int_{0^-}^t K^2_{m,n}(t-\tau)\ddot{q}_n(\tau)\mathrm{d}\tau \tag{8}$$

The deficiency functions $C^r_{m,n}(t)$ needed for our purpose and the apparent mass coefficients $D^r_{m,n}$ (which are zero for compressible flow) are defined such that

$$K^r_{m,n}(t)=K^r_{m,n}(\infty)-C^r_{m,n}(t)+D^r_{m,n}\delta(t) \tag{9}$$

where $K^r_{m,n}(\infty)$ are steady-state limits and $\delta(t)$ the Dirac delta function. In terms of these quantities, Eq. (8) assumes the form

$$\begin{aligned}K_{m,n}(t)&=K^1_{m,n}(\infty)q_n+[K^2_{m,n}(\infty)+D^1_{m,n}]\dot{q}_n+D^2_{m,n}\ddot{q}_n\\ &-\int_0^t C^1_{m,n}(t-\tau)\dot{q}_n(\tau)\mathrm{d}\tau-\int_{0^-}^t C^2_{m,n}(t-\tau)\ddot{q}_n(\tau)\mathrm{d}\tau\end{aligned} \tag{10}$$

This expression is more attractive than Eq. (8), since the deficiency functions vanish for $t\to\infty$.

A Common Deficiency Function

It is known that the Wagner function $\Phi(t)$, or rather the difference $1-\Phi(t)$, is sufficient for describing the deficiency functions for all of the deflection modes in the case of incompressible, two-dimensional flow. A similar result was found by this author[2] for a rectangular wing of aspect ratio 3. Calculated values of the normalized deficiency functions $C^r_{m,n}(t)/C^r_{m,n}(0)$ for this wing were almost equal for all of the deflection modes considered, i.e., independent of m, n, and r. There is reason to believe, therefore, that a single common function can be defined for each finite wing in incompressible flow in such a way that it approximates the normalized deficiency functions with sufficient accuracy for all of the important deflection modes.

The desired function may have the form

$$\varphi(t)=1/(1+t/T)^k \tag{11}$$

where $k=1$ and 3 for two- and three-dimensional flow, respectively, which yields the correct[5] behavior for $t\to\infty$. For two-dimensional flow and L equal to the semichord, it is further prescribed that $T=4$, which yields an approximation equivalent to that of Garrick.

The previous investigation for the rectangular wing produced the result $T=5.5$ for a reference length equal to the semichord.

Numerical Method

The numerical method employed herein is similar to the methods of Belotserkovskii,[11] Giesing,[12] and Djojodihardjo and Widnall,[13] but is restricted to a thin wing with an unswept trailing edge.

The boundary value problem for the velocity potential is solved in this method using doublet elements and approximating the potential jump by a constant within each element. The element grid and the wing that has been treated are shown in Fig. 1.

The elements are formed by equidistant orthogonal lines defined by

$$x=x_i=(i-1/4)2d_x, \qquad i=1,2,\ldots(i_s+1) \tag{12}$$

and

$$y=y_j=(j-1)2d_y, \qquad j=1,2,\ldots(j_s+1) \tag{13}$$

and the centers of the elements are $x=X_i=x_i+d_x$ and $y=Y_j=y_j+d_y$. The trailing edge and the wingtip are defined by $x=2$ and $y=b$, and the integers i_s and j_s and the lengths $2d_x$ and $2d_y$ of the element edges are such that $(2i_s+1)d_x=2$ and $(2j_s+1/2)d_y=b$. The definitions imply that the grid line at the wingtip lies at a distance $d_y/2$ inboard of the tip, and the centers of the upstream edges of the elements at the leading edge and the grid line at the trailing edge lie a distance $d_x/2$ downstream of the leading and trailing edges, respectively, which should yield accurate results for steady flow.[14]

The dimensionless semispan b, aspect ratio A, and taper ratio τ are related by $A=2b/(1+\tau)$.

The indicial motion starts at time $t=0$. At this time, the quantity $\Delta\phi_{\mu,\nu}$, which approximates the potential jump $\Delta\phi_n^r$ within the element with the center $(x,y)=(X_\mu, Y_\nu)$, can be solved from the equations

$$\sum_{\nu=1}^{j_s}\sum_{\mu=\nu}^{i_s} W_{i,j}^{\mu,\nu}\Delta\Phi_{\mu,\nu}(0)=w_n^r(X_i,Y_j)$$

$$j=1,2,\ldots j_s, \qquad i=j,j+1,\ldots i_s \tag{14}$$

where

$$w_n^r(x,y)=\frac{\partial h_n}{\partial x}, \qquad r=1$$

$$=h_n, \qquad r=2 \tag{15}$$

$$W_{i,j}^{\mu,\nu}=K(\mu-i,\nu-j)+sK(\mu-i,\nu+j-1) \tag{16}$$

$$K(i,j)=\lim_{z\to 0}\frac{-1}{4\pi}\int_{a_1}^{a_2}\mathrm{d}x\int_{b_1}^{b_2}\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\mathrm{d}y \tag{17}$$

$$R=(x^2+y^2+z^2)^{1/2} \tag{18}$$

$$a_2,a_1=(2i\pm 1)d_x \tag{19}$$

$$b_2,b_1=(2j\pm 1)d_y \tag{20}$$

and

$$s=1, \quad \text{if} \quad h_n(x,-y)=h_n(x,y)$$

$$=-1, \quad \text{if} \quad h_n(x,-y)=-h_n(x,y) \tag{21}$$

When the indicial motion starts, a vortex sheet begins to develop at the trailing edge of the wing. This sheet moves downstream relative to the wing with a velocity assumed to be equal to the freestream velocity. In order to calculate the effect of the vortex sheet, it is considered to be a planar surface with a potential jump. The normal velocity induced by this potential jump is calculated in the same way as the velocity induced by the jump across the wing. Hence, at time $t=t_k=k(2d_x)$, the potential jump approximations satisfy

$$\sum_{\nu=1}^{j_s}\sum_{\mu=\nu}^{i_s} W_{i,j}^{\mu,\nu}\Delta\phi_{\mu,\nu}(t_k)$$

$$=w_n^r(X_i,Y_j)-\sum_{\nu=1}^{j_s}\sum_{\mu=i_s+1}^{i_s+k} W_{i,j}^{\mu,\nu}\Delta\phi_{\mu,\nu}(t_k) \tag{22}$$

From Kelvin's theorem, which is automatically satisfied through the use of doublet elements, it follows that the doublet strength on an element which moves with the fluid is constant and equal to the strength generated when the element passes the trailing edge. This implies that

$$\Delta\phi_{\mu,\nu}(t_k)=\Delta\phi_{i_s,\nu}(t_{k-\mu+i_s}) \tag{23}$$

and the unknowns can be determined successively by solving Eq. (22) for $k=1,2,3,\ldots$.

The derivatives in Eq. (6) may be replaced by finite differences, yielding the approximate formula

$$\Delta p_{i,j}^{r,n}(t_{k+1})=-2[\Delta\phi_{i,j}(t_{k+1})-\Delta\phi_{i-1,j}(t_k)]/(2d_x) \tag{24}$$

for the pressure jump at time $t=t_{k+1}$. For time $t\approx 0$, the linearized Bernoulli equation gives

$$\Delta p_{i,j}^{r,n}(0)=-2\Delta\phi_{i,j}(0)\delta(t) \tag{25}$$

This may be used for calculation of the apparent mass coefficients.

The resulting approximate formula for the indicial coefficients is as follows:

$$K_{m,n}^r(t_k)=\frac{\Delta S}{S}\sum_{j=1}^{j_s}\sum_{i=j}^{i_s} h_m(x_i,Y_j)\Delta p_{i,j}^{r,n}(t_k) \tag{26}$$

where $\Delta S=4d_xd_y$.

Calculation for a Cropped Delta Wing

The method described was programmed in FORTRAN and run on the CRAY-1 computer. The present version of the program can treat a maximum of 500 elements on one wing half. It has been employed for a cropped delta wing with a trailing-edge flap (see Fig. 1). The wing has an aspect ratio of 2.4 and a taper ratio of 0.17, and the flap is bounded by the lines $y=20b/81$, $y=40b/81$, and $x=x_h=1.75$.

The program was run for $i_s=24$ and $j_s=20$ and for 100 time steps during which the wing moved slightly more than four root-chords.

The deflection modes treated were symmetric and defined by the equations

$$h_n=1, \qquad n=1$$

$$=g(y/b), \qquad n=2$$

$$=x, \qquad n=3$$

$$=xg(y/b), \qquad n=4$$

$$h_n=1 \quad \text{on the flap}, \qquad n=5$$

$$=0 \quad \text{off the flap},$$

$$h_n=x-x_h \quad \text{on the flap}, \qquad n=6$$

$$=0 \quad \text{off the flap}, \tag{27}$$

where

$$g(\eta) = 1.2\eta^2 - 0.2\eta^4 \tag{28}$$

Using Eq. (3), it can easily be seen that, for the modes considered,

$$\begin{aligned} K^1_{m,n}(t) &= 0, & n&=1 \\ &= 0, & n&=2 \\ &= K^2_{m,1}(t), & n&=3 \\ &= K^2_{m,2}(t), & n&=4 \\ &= 0, & n&=5 \\ &= K^2_{m,5}(t), & n&=6 \end{aligned} \tag{29}$$

It is thus sufficient to show results only for $K^2_{m,n}(t)$. The calculated values for the apparent mass coefficients $D^2_{m,n}$, the initial values $C^2_{m,n}(0)$ of the deficiency functions, and the steady-state limits $K^2_{m,n}(\infty)$ are given in Tables 1-3, while the main part of the results, the normalized deficiency functions, is plotted in Fig. 2.

The results for $C^2_{m,n}(t)$ and $K^2_{m,n}(\infty)$ probably are more accurate than those for $D^2_{m,n}$, because the grid has been better adapted for calculation of the former quantities than the latter. At the start of indicial motion, the potential jump near the trailing edge behaves like the square root of the distance from the edge. Hence, for calculation of apparent mass coefficients it would be better to use a grid with the most downstream line located a distance $d_x/2$ upstream instead of downstream of this edge. However, for the present study it is probably not important to calculate the apparent mass coefficients with high accuracy, since errors in them do not affect the normalized deficiency functions significantly.

The different circular symbols in Fig. 2 correspond to different weighting functions, i.e., different h_m in Eq. (7). Six values were computed corresponding to the modes considered, but, for clarity, only one value was plotted in the figure for each time. Since there are only three elements on the flap chord, high accuracy may not be obtained for the flap moment; therefore, the results for $m=6$ are not shown. The symbols near the curves in Fig. 2 represent results corresponding to the indicial normal velocity, which is defined by the value given for n.

All six curves in Fig. 2 represent the same function, i.e., the function φ [see Eq. (11)] for $k=3$ and $T=2.55$. The value obtained for T is a mean of values determined by the least-squares method for each of the normalized deficiency functions. The function fits the calculated values of the normalized deficiency functions including those not shown in the figure.

Comparing the result for T and the corresponding result for the rectangular wing previously discussed, T may be referred to the semispan divided by U. This yields $T=1.82$ and 1.83, i.e., almost the same result.

Approximation of Transfer Functions

The function φ can be used to approximate the aerodynamic transfer functions. To demonstrate this, we first apply Laplace transformation to Eq. (10). This yields the general expression

$$A_{m,n}(p) = K^1_{m,n}(\infty) + [K^2_{m,n}(\infty) + D^1_{m,n}]p + D^2_{m,n}p^2 - [pL\{C^1_{m,n},p\} + p^2L\{C^2_{m,n},p\}] \tag{30}$$

Table 1 Apparent mass coefficients $D^2_{m,n}$

	$n=$					
m	1	2	3	4	5	6
1	1.6339	0.3031	2.1752	0.4625	0.1088	0.0154
2	0.3031	0.1064	0.4515	0.1722	0.0253	0.0036
3	2.1082	0.4391	3.0015	0.6911	0.1788	0.0256
4	0.4501	0.1678	0.6916	0.2767	0.0430	0.0062
5	0.0110	0.0026	0.0187	0.0045	0.0047	0.0008

Table 2 Initial values $C^2_{m,n}(0)$

	$n=$					
m	1	2	3	4	5	6
1	0.5255	0.1324	0.9158	0.2396	0.1417	0.0266
2	0.1770	0.0468	0.3089	0.0849	0.0476	0.0089
3	0.6084	0.1562	1.0607	0.2830	0.1640	0.0307
4	0.2430	0.0650	0.4241	0.1181	0.0653	0.0122
5	0.00110	0.00028	0.00191	0.00052	0.00030	0.000055

Table 3 Steady-state limits $K^2_{m,n}(\infty)$

	$n=$					
m	1	2	3	4	5	6
1	2.7193	0.6979	4.7415	1.2650	0.7242	0.1357
2	0.7464	0.2748	1.3174	0.5071	0.1867	0.0349
3	2.8609	0.8704	5.6658	1.6667	1.0842	0.2081
4	0.9871	0.3875	1.8293	0.7408	0.2928	0.0559
5	0.0049	0.0016	0.0173	0.0049	0.0138	0.0047

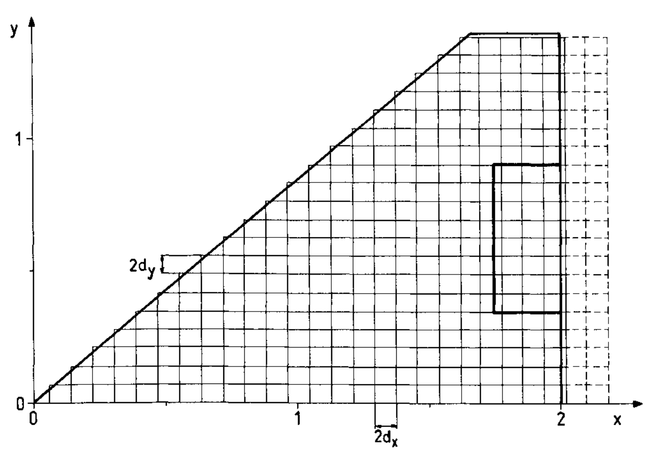

Fig. 1 Doublet elements representing wing with flap.

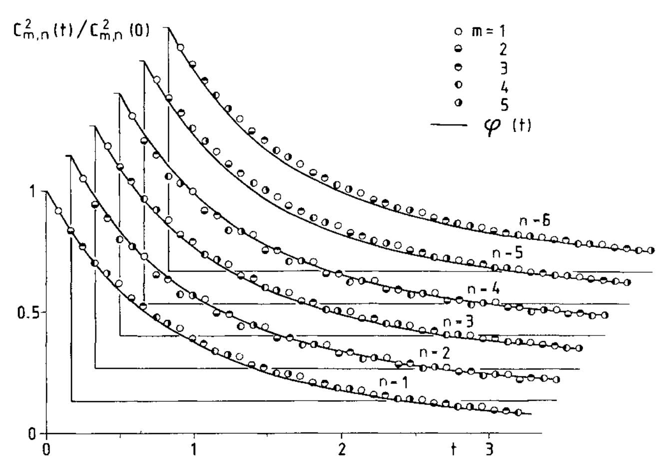

Fig. 2 Normalized results for deficiency functions and the function $\varphi = 1/(1+t/T)^3$ for $T=2.55$.

for the aerodynamic transfer functions, and the relation

$$L\{K_{m,n},p\} = A_{m,n}(p)L\{q_n,p\} - D^2_{m,n}\dot{q}_n(0) \tag{31}$$

for the transform of the aerodynamic coefficients. The quantity p is the dimensionless Laplace transform parameter. Since the motion is continuous, the initial values $q_n(0)$ of the generalized coordinates are zero and do not contribute to the transform.

The transfer functions are identical to the ordinary unsteady aerodynamic coefficients. This is easily seen by using Eq. (10) to calculate the limit of $K_{m,n}(t)$ for $t\to\infty$ for $q_n = \exp(i\omega t)$. The result $K_{m,n}(t)\to A_{m,n}(i\omega)\exp(i\omega t)$ shows that to obtain the transfer functions from the unsteady coefficients it is simply sufficient to replace the argument $i\omega$ by p. Also note that the imaginary unit i enters only via p so that $A_{m,n}(p)$ is real for points on the positive real axis.

Expression (30) corresponds to the formulas of Küssner and Theodorsen. It reduces, namely, for plunge and pitch about 50% chord to

$$\frac{1}{\pi}[A_{m,n}] = \begin{bmatrix} 0, & 2 \\ 0, & -1 \end{bmatrix} + \begin{bmatrix} 2, & 2 \\ -1, & 0 \end{bmatrix} p + \begin{bmatrix} 1, & 0 \\ 0, & 1/8 \end{bmatrix} p^2$$

$$- \left\{ \begin{bmatrix} 0, & 2 \\ 0, & -1 \end{bmatrix} + p \begin{bmatrix} 2, & 1 \\ -1, & -1/2 \end{bmatrix} \right\} pL\{1-\Phi,p\} \tag{32}$$

which agrees with those formulas since

$$pL\{1-\Phi,p\} = 1 - C(-ip) \tag{33}$$

$C(\omega)$ is the Theodorsen function.

Approximating the deficiency functions in Eq. (30) by means of $\varphi(t)$, we now get

$$A_{m,n}(p) = K^1_{m,n}(\infty) + [K^2_{m,n}(\infty) + D^1_{m,n}]p + D^2_{m,n}p^2$$

$$- [C^1_{m,n}(0) + pC^2_{m,n}(0)]pL\{\varphi,p\} \tag{34}$$

For $k=1$ and $T=4$, the transform $pL\{\varphi,p\}$ is approximately equal to $2(1-C(-ip))$, and for $k=3$ and if T is determined properly, the function $1-\frac{1}{2}pL\{\varphi,p\}$ is a generalized counterpart to $C(-ip)$. For calculation of $pL\{\varphi,p\}$, the following formulas may be used:

$$pL\{\varphi,p\} = pTF_k(pT) \tag{35}$$

$$F_k(p) = \int_0^\infty e^{-pt}(1+t)^{-k}\mathrm{d}t$$

$$= (1 - pF_{k-1}(p))/(k-1) \tag{36}$$

$$F_1(p) = e^p E_1(p) \tag{37}$$

and for the exponential integral $E_1(p)$, the following expansion[15] may be used:

$$E_1(p) = -\gamma - \ln(p) - \sum_{n=1}^{\infty} (-1)^n p^n$$

$$\div (n(n!)) \quad |\arg(p)| < \pi \tag{38}$$

where γ is Euler's constant, 0.577215....

The function $pL\{\varphi,p\}$ contains a logarithmic contribution of the form $p\ln(p)$ for $k=1$ and a contribution of the form $p^3\ln(p)$ for $k=3$, which agrees with the known[5] results. Due to the logarithmic contributions, the function is discontinuous on the negative real axis, similar to the Theodorsen function. This appears in Fig. 3, where the counterpart $1-(1/2)pTF_3(pT)$ to $C(-ip)$ is plotted.

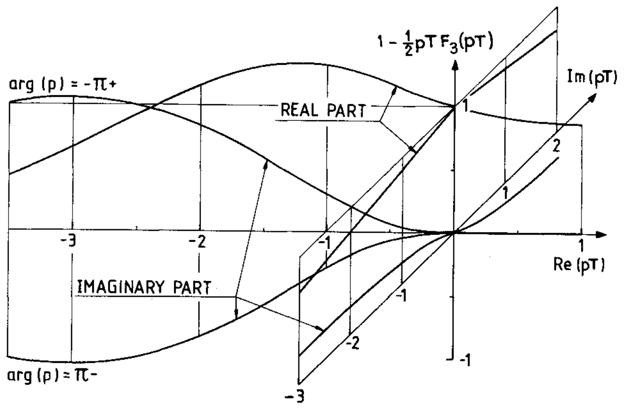

Fig. 3 Generalized Theodorsen function.

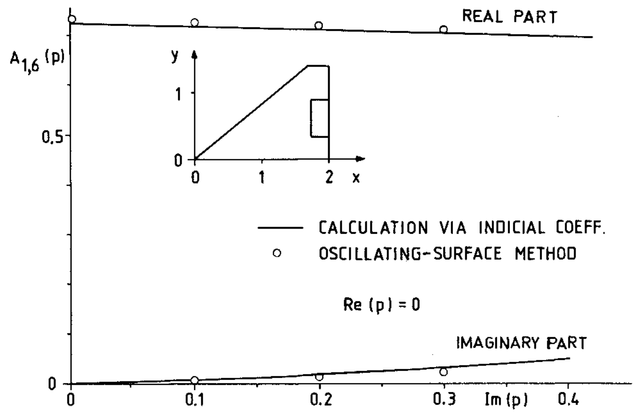

Fig. 4 Comparison of results for transfer function for lift due to flap rotation.

Further calculations for a set of planforms may yield sufficient knowledge so that a suitable value can be selected for the characteristic time T for any wing. To obtain the desired transfer function approximation in a flutter program, it would then suffice to calculate the steady-state limits $K^r_{m,n}(\infty)$, the apparent mass coefficients, and the initial value of the deficiency functions.

Another perhaps more practical way of using the function φ is to calculate the transfer functions $A_{m,n}(p)$ for some discrete values of p by an oscillating-surface program and determine the constants in expression (34) on the basis of the calculated values. Since φ functions properly in the time domain, it is expected that expression (34) functions properly in the p domain so that it may suffice to calculate only a few values of $A_{m,n}(p)$ to determine the unknown constants.

An Independent Check

The values given in Tables 1-3 for $D^2_{m,n}$, $C^2_{m,n}(0)$, and $K_{m,n}(\infty)$ were used to calculate $A_{m,n}(p)$ for a few imaginary values of p using Eqs. (34-38) for $k=3$ and $T=2.55$. In Fig. 4, the results are compared to corresponding results from a new oscillating-surface program based on a doublet element method. The jump in the advanced velocity potential[16] is approximated in this method by constants on small trapezoidal wing elements. Two hundred elements formed by constant percent chord lines and streamwise lines were used on each wing-half.

It is seen from Fig. 3, which shows results for the transfer function for lift due to rigid flap rotation, that the results are in close agreement and, thus, the two methods yield satisfactory accuracy.

Comparison with Jones' Approximations

Due to the correct behavior of the function $\varphi(t)$ for $t \to \infty$ and of the transform $L\{\varphi,p\}$ in the neigborhood of the negative real axis of the p plane, it is expected that approximation of the aerodynamic transfer functions by means of $L\{\varphi,p\}$ yields more accurate eigenvalues for stable aeroelastic modes than the Jones' approximations[7,8] or the related approximations of Vepa.[6] The deviations increase for eigenvalues approaching the negative real axis, which is due to the completely different behavior of the latter approximations in this part of the plane.

Thus, one might think that approximations of the type proposed herein would yield more accurate solutions to the equations of motion than approximations of the Jones' type, but this need not be the case. The different approximations yield different eigenvalues but not necessarily different responses to a given exciting force. Both approximations may be accurate, i.e., on the imaginary axis, which ordinarily satisfies the condition for the integration path of the Laplace inversion integral.

Whether the approximation yields accurate eigenvalues or not, a response calculation can be performed either by evaluating the inversion integral by integration on the imaginary axis or by summation of residues at all poles (eigenvalues) to the right of an arbitrary integration path (which separates these poles from other less interesting singularities to the left) and by evaluating and adding the remaining integral on this path to the residue sum. If accurately performed, both calculations yield (due to Cauchy's theorem) the same result irrespective of the accuracy of the approximation (which is assumed to be an analytic function of p).

The latter calculation is attractive, since it may be possible to choose an integration path such that the integral on it becomes unimportant or easy to evaluate. For approximations of the Jones' type, there are paths on which the integral is zero.

A more complete treatment of the approximation problem, which must consider not only incompressible flow but compressible subsonic and supersonic flow as well, is not possible in this context. Some results of approximations for compressible flow are given in Ref. 17.

Conclusions

Indicial aerodynamic coefficients and deficiency functions have been calculated for typical deflection modes of a cropped delta wing with a trailing-edge flap in incompressible flow.

The results for the deficiency functions, when normalized, were found to be closely approximated by the function $\varphi = 1/(1+t/T)^3$ for a suitable value of the characteristic time T.

The normalized results for the deficiency functions for different deflection modes were found to be almost equal, which implies that the function φ represents all deficiency functions with good accuracy for a single value of T.

Since φ represents the normalized results for the deficiency functions with good accuracy, the Laplace transform of φ for a single value of T yields a generalized counterpart to the Theodorsen function. The counterpart can be used to approximate the transfer functions for different typical deflection modes.

For the wing treated herein, with an aspect ratio of 2.4 and a taper ratio of 0.17, the value obtained for T, when measured in semispans traveled, was 1.82. This result is very close to that obtained for T for a rectangular wing of aspect ratio 3 in an earlier investigation.

In order to gain further experience and data for T, it is desirable to perform calculations for some other planforms.

Acknowledgment

The author is indebted to the Swedish Defence Administration and Saab Scania AB for financing this investigation.

References

[1]Stark, V.J.E., "A Flutter Eigenvalue Program Based on the Newton-Raphson Method," *AIAA Journal*, Vol. 22, July 1984, pp. 993-995.

[2]Stark, V.J.E., "General Equations of Motion for an Elastic Wing and Method of Solution," *AIAA Journal*, Vol. 22, Aug. 1984, pp. 1146-1153.

[3]Garrick, I.E., "On Some Reciprocal Relations in the Theory of Nonstationary Flows," NACA Rept. 629, 1938.

[4]Lomax, H., Heaslet, M.A., Fuller, F.B., and Sluder, L, "Two- and Three-Dimensional Unsteady Lift Problems in High-Speed Flight," NACA Rept. 1077, 1952.

[5]Milne, R.D., "Asymptotic Solutions of Linear Stationary Integro-Differential Equations," ARC R&M 3548, 1968.

[6]Vepa, R., "On the Use of Padé Approximants to Represent Unsteady Aerodynamic Loads for Arbitrarily Small Motions of Wings," AIAA Paper 76-17, 1976.

[7]Jones, R.T., "The Unsteady Lift of a Wing of Finite Aspect Ratio," NACA Rept. 681, 1940.

[8]Jones, W.P., "Aerodynamic Forces on Wings in Non-uniform Motion," ARC R&M 2117, 1945.

[9]Wuytack, L., "On the Conditioning of the Padé Approximation Problem," *Proceedings, Padé Approximation and Its Applications*, Amsterdam, 1980, published by Springer-Verlag, Berlin, 1981, pp. 78-89.

[10]Rodden, W.P. and Bellinger, E.D. "Aerodynamic Lag Functions, Divergence, and the British Flutter Method," *Journal of Aircraft*, Vol. 19, July 1982, pp. 596-598.

[11]Belotserkovskii, S.M., "Calculating the Effect of Gusts on Arbitrary Thin Wing," *Mekhanika Zidkosti i Gaza*, Vol. 1, No. 1, 1967, pp. 34-40; also *Translations in Fluid Dynamics*, Vol. 1, No. 1, 1967.

[12]Giesing, J.P., "Nonlinear Two-Dimensional Unsteady Potential Flow with Lift," *Journal of Aircraft*, Vol. 5, March-April 1968, pp. 135-143.

[13]Djojodihardjo, R.H. and Widnall, S.E., "A Numerical Method for the Calculation of Nonlinear Unsteady Lifting Potential Flow Problems," AIAA Paper 69-23, 1969.

[14]Wieghardt, K., "Über die Auftriebsverteilung des einfachen Rechteckflügels über die Tiefe," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 19, No. 5, 1939, pp. 257-270.

[15]Gautschi, W. and Cahill, W.F., "Exponential Integral and Related Functions," *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I.A. Stegun, National Bureau of Standards, Applied Mathematics Series 55, 9th Printing, 1970, pp. 227-251.

[16]Landahl, M.T. and Stark, V.J.E., "Numerical Lifting-Surface Theory—Problems and Progress," *AIAA Journal*, Vol. 6, Nov. 1968, pp. 2049-2060.

[17]Stark, V.J.E., "Flutter Calculation by a New Program," *Proceedings of the Second International Symposium on Aeroelasticity and Structural Dynamics*, April 1985, Aachen, FRG.